BCA Part I

S.S. College. J'Bad.
Topic – Real Analysis (Convergence of Series)
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1. Sequence of Partial Sums of a Series, Convergence of Series :→

Let $u_1 + u_2 + \cdots + u_n + \cdots$ be an infinite series.

S_n = Sum of First n Terms

i.e $S_n = u_1 + u_2 + \cdots + u_n$

then the sequence $\{S_n\}$ is called the Sequence of Partial Sums of the Given Series.

If $\lim_{n\to\infty} S_n = S$ then Series is called Convergent.

If $\lim_{n\to\infty} S_n = +\infty$ or $= -\infty$ then the Series is called Divergent.

Oscillatory Series – If $\{S_n\}$ tends to no definite limit whether finite or infinite as $n\to\infty$, then the series is said to oscillate.

We say that the series oscillates finitely or infinitely according as S_n oscillates between finite limits or between $+\infty$ and $-\infty$

* If an infinite Series Σu_n is convergent, then $u_n \to 0$ as $n\to\infty$

this is Only Necessary Condition but is not Sufficient.

Ex :→ Consider the series

$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$

Here $u_n = \frac{1}{n}$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

But the series is divergent.

for $\frac{1}{3}+\frac{1}{4} > \frac{1}{4}+\frac{1}{4} = \frac{1}{2}$

$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{1}{2}$ and so on.

$\therefore 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ \cdots$

$> \left(1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots \text{to} \infty\right) \to \infty$

and is therefore divergent.

Hence $\lim_{n\to\infty} u_n = 0$ But the series is not ~~ ~~ convergent.

Thus we may say

$\lim_{n\to\infty} u_n = 0$ is necessary condition for convergent of a series but not sufficient condition.

Convergence of a Geometric Series :→

The geometric series

$1+r+r^2+r^3+ \cdots$ to ∞

is convergent if $|r|<1$

is divergent if $r \geq 1$

oscillates if $r \leq -1$

*. Convergence of series of +ve terms

1. A series Σu_n of positive term is either convergent or divergent.

2. If Σu_n and Σv_n are two series

of +ve Terms, Convergent to sum S and S' respectively, then the series $\Sigma u_n + v_n$

and $\Sigma u_n - v_n$ are also Convergent to the sums $S+S'$ and $S-S'$ respectively

* A series Σu_n of +ve Terms Converges if and only if there exists a finite number K such that
$$u_1 + u_2 + \cdots + u_n < K \text{ for all } n$$

* If the Terms of a Convergent Series of positive terms are rearranged, the series remains convergent and its sum remain same.

*. Auxiliary Series :→
the Infinite Series $\Sigma \frac{1}{n^p}$

is (i) Convergent if $p > 1$
(ii) divergent if $p \leq 1$

* State and Prove Abel's theorem or Pringsheim's theorem –

Statement – If Σu_n is Convergent Series of +ve decreasing terms, then $\lim_{n \to \infty} n u_n = 0$

Proof :→ Since Σu_n is a ~~convergent~~ series of Positive decreasing terms

Therefore $u_{n+1} \geq u_{2n}$

$u_{n+2} \geq u_{2n}$

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$u_{2n} = u_{2n}$.

On Adding we get-

$$u_{n+1} + u_{n+2} + u_{n+3} + \dots + u_{2n} \geq n u_{2n}.$$

Since $\sum u_n$ is convergent, by the General Condition for convergence,

$u_{n+1} + u_{n+2} + u_{n+3} + \dots + u_{2n} \to 0$ as $n \to \infty$

$$\therefore \lim_{n\to\infty} n u_n = 0$$

$$\Rightarrow \lim_{n\to\infty} 2n u_n = 0 \quad \text{——— (1)}$$

Therefore, Abel's theorem is proved when n is even.

Again $\lim_{n\to\infty} \frac{2n+1}{2n} = \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right) = 1$ ——— (2)

Now, $u_{2n+1} \leq u_{2n}$

$$(2n+1) u_{2n+1} \leq \frac{2n+1}{2n} 2n u_{2n}$$

$$\Rightarrow (2n+1) u_{2n+1} \leq \left(1 + \frac{1}{2n}\right)(2n u_{2n}.)$$

$$\Rightarrow \lim_{n\to\infty} \left[(2n+1) u_{2n+1}\right] \leq \lim_{n\to\infty} \left(1 + \frac{1}{2n}\right) \left[\lim_{n\to\infty} (2n u_{2n})\right]$$

$$\Rightarrow \lim_{n\to\infty} \left[(2n+1) u_{2n+1}\right] = 0 \quad \text{From (1) and (2)}$$

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* Comparison Test →

Form I — (i) If $u_n \le K v_n$, for all values of n, where $n > 0$ and K is a fixed positive number and Σv_n is convergent, then Σu_n is also convergent.

(ii) If $u_n \geq K v_n$, for all values of n, where $n > 0$ and K is a fixed positive number and Σv_n is divergent, then Σu_n is also divergent.

Form II → If $\lim_{n\to\infty} \frac{u_n}{v_n}$ = a finite and non-zero number l, then Σu_n and Σv_n are either both convergent or both divergent.

Problem → Test for convergence of the series

$$1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \cdots + \frac{1+n}{1+n^2} + \cdots$$

or

Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1+n}{1+n^2}$$

Here $u_n = \frac{1+n}{1+n^2}$

Compare the given series Σu_n with the ~~Auxill~~ Auxiliary series Σv_n

where $v_n = \frac{n}{n^2} = \frac{1}{n}$

then $\frac{u_n}{v_n} = \frac{n+n^2}{1+n^2} = \frac{n^2(1+1/n)}{n^2(1+1/n^2)} = \frac{1+1/n}{1+1/n^2}$

Therefore

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n}+1}{\frac{1}{n^2}+1} = 1 \text{ Which}$$

is finite and $\neq 0$.

Hence By Comparision Test

Σu_n and Σv_n

are either both convergent or both divergent.